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SQUARING THE MAGIC SQUARES OF ORDER 4

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Abstract

In this paper, we present the problem of counting magic squares and we focus on the case of multiplicative magic squares of order 4. We give the exact number of normal multiplicative magic squares of order 4 with an original and complete proof, pointing out the role of the action of the symmetric group. Moreover, we provide a new representation for magic squares of order 4. Such representation allows the construction of magic squares in a very simple way, using essentially only five particular 4×4 matrices.

1. Introduction

A magic square is defined as an $n \times n$ matrix of integers, where the sum of the numbers in each *line* (i.e., in each row, in each column, and in each diagonal) is the same. Magic squares have a very rich history (see,

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e.g., the beautiful book of Descombes [6]) and they have many generalizations. There are a lot of contributes to their theory from several different fields (see, e.g., the classical book of Andrews [1]).

A very difficult job is counting how many magic squares there are for a given order and a given line sum, but if we relax the requirements and we ask for the so-called semi-magic squares (i.e., magic squares without the condition on the diagonals), the counting is easier. For order 3, Bona [2] has found that the number of semi-magic squares is

$$\binom{r+4}{4}+\binom{r+3}{4}+\binom{r+2}{4},$$

where r is the line sum (let us observe that in [2] semi-magic squares are called *magic squares*). For a general theory of counting semi-magic squares, you can see ([3], Chapter 9).

The classic magic squares, which are called *normal*, are $n \times n$ matrices, whose entries consist of the numbers $0, 1, \ldots, n^2 - 1$ and in each line, the sum of the numbers is the same. Their number is known for the orders 3, 4, and 5. The case of order 4 is especially interesting. There are exactly 880 normal magic squares of order 4 (up to symmetries of the square, i.e., in total they are $880 \times 8 = 7040$). They were enumerated for the first time in 1693 by Frenicle de Bessy. The Frenicle method was analytically expanded and completed by Bondi and Ollerenshaw [4]. Of course, one can think to different kinds of magic squares, only changing the operation. Then a *multiplicative magic square* is an $n \times n$ matrix of integers, where the product of the numbers in each line is the same. This is not a new idea, for example, multiplicative semi-magic squares of order 3 are studied in [7]. There are several different methods to construct multiplicative magic squares. The most obvious method is to transform an additive magic square $A = (a_{ij})$ into $M = (b^{a_{ij}})$, for any base b. More intriguing ways are explained in [6] (multiplicative magic squares are usually constructed by using orthogonal latin squares or geometric progressions).

In [9], the problem of finding the exact number of multiplicative magic squares of order 4 has been posed. In order to solve such question, in [8], a correspondence between additive and multiplicative magic squares is proposed. The authors claimed that such function is an isomorphism even though it is only an injective morphism, as pointed out in [10]. Since such revision is only a note, the above problem is not completely clarified and the correct number of multiplicative magic squares is given without a rigorous proof. In the next section, we clarify the above questions, proposing an original and complete proof. Moreover, in the last section, a novel representation for some magic squares is proposed, allowing an easy way to construct them.

2. Counting Multiplicative Magic Squares of Order 4

A *normal multiplicative magic square* $n \times n$ is created using n primes $p_1, \ldots, p_n$. All the divisors of $k = p_1 \cdots p_n$ must appear once time in the matrix and in each line the product is the same. The magic constant is k^2 and it is well-known that in a normal multiplicative magic square, in each line, any prime compares with power one and exactly two times. For example, a normal multiplicative magic square of order 4 is

$$\begin{pmatrix} 1 & p_3 \cdot p_4 & p_1 \cdot p_2 & p_1 \cdot p_2 \cdot p_3 \cdot p_4 \\ p_1 \cdot p_3 \cdot p_4 & p_1 \cdot p_2 \cdot p_3 & p_4 & p_2 \\ p_1 \cdot p_2 \cdot p_4 & p_1 & p_2 \cdot p_3 \cdot p_4 & p_3 \\ p_2 \cdot p_3 & p_2 \cdot p_4 & p_1 \cdot p_3 & p_1 \cdot p_4 \end{pmatrix}.$$

In the following, we use $\mathbb{M}_{k,n}$ to indicate the set of normal multiplicative magic squares $n \times n$ with magic constant k^2. A hard problem to solve is to find the order of $\mathbb{M}_{k,4}$, as posed in [9]. In order to answer to this question, $\mathbb{M}_{k,4}$ is made in correspondence with the set of *normal additive magic squares* 4×4 [8].

A *normal additive magic square* is a matrix $n \times n$, whose entries consist of the numbers $0, 1, \ldots, n^2 - 1$ and in each line, the sum of the numbers is the same. The magic constant is $\frac{n(n^2-1)}{2}$. From now on, we name $\mathbb{A}_n$ the set of normal additive magic squares $n \times n$.

It is well-known that the order of $\mathbb{A}_4$ is 7040, or 880 up to symmetries of the square [4]. Can we find a similar result for $\mathbb{M}_{k,4}$?

In [8], the following correspondence is defined:

$$f : \mathbb{M}_{k,n} \to \mathbb{A}_{2^{n/2}}.$$

If f is bijective, $|\mathbb{M}_{k,4}|$ is determined. The function f maps $M = (m_{ij}) \in \mathbb{M}_{k,n}$ into $A = (a_{ij}) \in \mathbb{A}_{2^{n/2}}$. The element m_{ij} is made in correspondence with a_{ij}, where a_{ij} is the base 10 number , between 0 and $n^2 - 1$, whose base 2 expression is the n-string associated to m_{ij}. This string is constructed as follows: arrange the n prime numbers in ascending order; if p_k is a factor of m_{ij} place a 1 in the k-th position of the string; if not, place a 0.

Such a function f is surely injective, but unfortunately, it is not surjective and the order of $\mathbb{M}_{k,4}$ is not $|\mathbb{A}_4| = 7040$. Indeed, we can consider the group S_4, which permutes the four primes involved in $\mathbb{M}_{k,4}$.

Theorem 1. *The order of* S_4 *divides the order of* $\mathbb{M}_{k,4}$.

Proof. Surely, when a permutation $\rho \in S_4$ acts on $M \in \mathbb{M}_{k,4}$, then $\rho(M)$ is still an element in $\mathbb{M}_{k,4}$. Furthermore, if ρ is not the identity, it does not fix any element of $\mathbb{M}_{k,4}$, i.e., any orbit has 24 elements. Therefore by Burnside lemma, $|S_4|$ must be a divisor of $|\mathbb{M}_{k,4}|$. □

As immediate consequence of the previous theorem, 7040 can not be the order of $\mathbb{M}_{k,4}$, since 24 does not divide 7040. The correct answer is given in the following theorem:

Theorem 2. *The order of* $\mathbb{M}_{k,4}$ *is* 4224.

Proof. If we consider $A \in \mathbb{A}_4$ and we write its entries in base 4, then in each line, the sum of the digits in the first position multiplied by 4 and added to the sum of the digits in the second position must yield 30. The only possible combinations are

$$30 = 7 \times 4 + 2 = 6 \times 4 + 6 = 5 \times 4 + 10.$$

So, if A has a line in which the digits in the second position have sum 2, we are in the only two situations

$$\begin{array}{cccc} \cdot 0 & \cdot 0 & \cdot 1 & \cdot 1 \\ \cdot 0 & \cdot 0 & \cdot 0 & \cdot 2, \end{array}$$

where $\cdot$ indicates the first position of the number in base 4. Considering the representation in base 2 of these strings, we have one of the following situations:

$$\begin{array}{cccc} \cdot\cdot 00 & \cdot\cdot 00 & \cdot\cdot 01 & \cdot\cdot 01 \\ \cdot\cdot 00 & \cdot\cdot 00 & \cdot\cdot 00 & \cdot\cdot 10. \end{array}$$

Now if we try to create $M \in \mathbb{M}_{k,4}$, using the inverse of f, if A has a line which presents one of these situations, we can not have a normal multiplicative magic square. In fact, if we are in the first situation, for example, then the prime p_3 will not appear in this line. Similarly, if a line of A has digits in the second position with sum 10, the possible cases are

$$\begin{array}{cccc} \cdot 3 & \cdot 3 & \cdot 3 & \cdot 1 \\ \cdot 2 & \cdot 2 & \cdot 3 & \cdot 3, \end{array}$$

which correspond in base 2 to the strings

$$\begin{array}{cccc} \cdot\cdot 11 & \cdot\cdot 11 & \cdot\cdot 11 & \cdot\cdot 01 \\ \cdot\cdot 10 & \cdot\cdot 10 & \cdot\cdot 11 & \cdot\cdot 11. \end{array}$$

Once more, if we try, we fail to obtain $M \in \mathbb{M}_{k,4}$.

Finally, if A has a line whose digits in the second position have sum 6, we have the possibilities

$$\begin{array}{cccc} \cdot 0 & \cdot 1 & \cdot 2 & \cdot 3 \\ \cdot 0 & \cdot 0 & \cdot 3 & \cdot 3 \\ \cdot 1 & \cdot 1 & \cdot 2 & \cdot 2 \\ \cdot 0 & \cdot 2 & \cdot 2 & \cdot 2 \\ \cdot 1 & \cdot 1 & \cdot 1 & \cdot 3. \end{array}$$

With similar arguments as the ones used before, it is easy to see that the last two situations do not allow to obtain a normal multiplicative magic square. Therefore, the squares $A \in \mathbb{A}_4$ from which we obtain a normal multiplicative magic square are only those with lines, whose entries in base 4 have digits in the first and second position composed by strings

$$\begin{array}{cccc} \cdot 0 & \cdot 1 & \cdot 2 & \cdot 3 \\ \cdot 0 & \cdot 0 & \cdot 3 & \cdot 3 \\ \cdot 1 & \cdot 1 & \cdot 2 & \cdot 2. \end{array}$$

In [4], all the squares in $\mathbb{A}_4$ are classified. The squares in $\mathbb{A}_4$ generated using only these strings, i.e., which correspond to squares in $\mathbb{M}_{k,4}$, are only those in category one ([4], p. 510). They are exactly 528 unless of symmetries of the square. Therefore, our arguments and the injectivity of f allow us to conclude that the order of $\mathbb{M}_{k,4}$ is $528 \times 8 = 4224$. □

3. A New Representation for Magic Squares

In this section, we see a representation for normal additive magic squares, which correspond to normal multiplicative ones.

In [5], a lower bound for the distance between the maximal and minimal element in a multiplicative magic square is given. In order to do that, the relation between additive and multiplicative magic squares is highlighted, recalling that a multiplicative magic square can be factorized as $\prod_i p_i^{A_i}$, where A_i's are additive magic squares. Moreover, focusing on magic squares of order 4, in [5], the Hilbert basis (composed by 20 magic squares) for such magic squares is explicited. However, such representation and the relation between additive and multiplicative magic squares are not really manageable in order to construct additive and multiplicative magic squares of order 4. Here, the proposed representation allows to determine all (and not only) the normal additive magic squares of order 4, which corresponds to normal multiplicative magic squares. In this way, they can all be easily constructed essentially using only 5 basic matrices.

We consider $M = (m_{ij}) \in \mathbb{M}_{k,4}$, by means of f, we have the correspondent $A = (a_{ij}) \in \mathbb{A}_4$ and we consider its entries in base 2. Now, we decompose A into four matrices A_1, A_2, A_3, A_4, whose entries are only 0 or 1, so that the entries in position ij of the matrices A_1, A_2, A_3, A_4 form the string a_{ij}.

Example 1. From the normal multiplicative magic square

$$\begin{pmatrix} p_1p_2p_3 & p_3p_4 & 1 & p_1p_2p_4 \\ p_2p_4 & p_1 & p_1p_3p_4 & p_2p_3 \\ p_1p_4 & p_2 & p_2p_3p_4 & p_1p_3 \\ p_3 & p_1p_2p_3p_4 & p_1p_2 & p_4 \end{pmatrix},$$

using f, we obtain the normal additive magic square

$$\begin{pmatrix} 1110 & 0011 & 0000 & 1101 \\ 0101 & 1000 & 1011 & 0110 \\ 1001 & 0100 & 0111 & 1010 \\ 0010 & 1111 & 1100 & 0001 \end{pmatrix} = \begin{pmatrix} 14 & 3 & 0 & 13 \\ 5 & 8 & 11 & 6 \\ 9 & 4 & 7 & 10 \\ 2 & 15 & 12 & 1 \end{pmatrix},$$

and it can be decomposed into

$$8\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$+ 2\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Since A is derived from a matrix in $\mathbb{M}_{k,4}$, by means of f, the matrices A_1, A_2, A_3, A_4 are all and only those having in each line exactly two ones. We call *forms* these matrices, which allow us to construct any magic square in $\mathbb{A}_4$ corresponding to magic squares in $\mathbb{M}_{k,4}$. These forms are exactly 16.

Theorem 3. *There are* 16 *different matrices* 4×4, *with entries* 0 *or* 1, *such that in each line there are exactly two ones.*

Proof. The strings that we can use to generate these forms are only six

$$1100, \quad 1010, \quad 1001, \quad 0110, \quad 0101, \quad 0011.$$

If we choose as a diagonal a string with different extremes, then we have only two possible different forms. For example,

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{pmatrix} \to \begin{pmatrix} 1 & \cdot & \cdot & 0 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{pmatrix} \to \begin{pmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

On the other hand, if we choose as a diagonal a string with same extremes, then we have four possible forms. Since, we have four strings with different extremes and two strings with same extremes, our forms are $4 \cdot 2 + 2 \cdot 4 = 16$. □

From these 16 forms, we can individuate five *fundamental forms*:

$$A_0 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \; B_0 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \; D_0 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$E_0 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

All the remaining can be found acting on these five forms with the group of symmetries of the square D_8. Combining any four forms, as we have done in the previous example, the resulting matrix is always an additive magic square, not necessarily normal, and it corresponds to a multiplicative one.

Theorem 4. *If A_1, A_2, A_3, A_4 are fundamental forms, or matrices obtained from fundamental forms through the action of D_8, then*

$$8A_1 + 4A_2 + 2A_3 + A_4$$

is always an additive magic square, which corresponds to a multiplicative magic square.

Given a magic square $8A_1 + 4A_2 + 2A_3 + A_4$ constructed using our forms, we can consider all the permutations of the forms A_1, A_2, A_3, A_4. After a permutation, we have still a magic square. Thus, we can use the group S_4 over our forms.

Example 2. We consider

$$8\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$+2\begin{pmatrix}0&1&1&0\\0&1&1&0\\1&0&0&1\\1&0&0&1\end{pmatrix}+\begin{pmatrix}1&1&0&0\\0&0&1&1\\0&0&1&1\\1&1&0&0\end{pmatrix},$$

which corresponds to

$$\begin{pmatrix}5&3&14&8\\12&10&7&1\\2&4&9&15\\11&13&0&6\end{pmatrix}.$$

Permuting the positions of the forms, we obtain

$$8\begin{pmatrix}1&1&0&0\\0&0&1&1\\0&0&1&1\\1&1&0&0\end{pmatrix}+4\begin{pmatrix}0&0&1&1\\1&1&0&0\\0&0&1&1\\1&1&0&0\end{pmatrix}$$

$$+2\begin{pmatrix}1&0&1&0\\1&0&1&0\\0&1&0&1\\0&1&0&1\end{pmatrix}+\begin{pmatrix}0&1&1&0\\0&1&1&0\\1&0&0&1\\1&0&0&1\end{pmatrix},$$

which is equal to

$$\begin{pmatrix} 10 & 9 & 7 & 4 \\ 6 & 5 & 11 & 8 \\ 1 & 2 & 12 & 15 \\ 13 & 14 & 0 & 3 \end{pmatrix}.$$

We have seen that using our forms we obtain always an additive magic square, but it is not necessarily *normal*. Finally, let us see how to utilize the forms in order to generate all and only the normal additive magic squares corresponding to normal multiplicative magic squares. The orbit of the fundamental forms with respect to the action of D_8 are

$$A = \{A_0, A_1\},$$

$$B = \{B_0, B_1, B_2, B_3\},$$

$$C = \{C_0, C_1, C_2, C_3\},$$

$$D = \{D_0, D_1, D_2, D_3\},$$

$$E = \{E_0, E_1\}.$$

We easily observe that

$$A_0 + A_1 = E_0 + E_1 = \mathbb{U} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Furthermore, for any form in the orbits $B, C, D,$ there is another form, in the same orbit, such that their sum is $\mathbb{U}$.

We call *class* the set $(A, B, C, D),$ whose elements are all the magic square obtained combining the forms belonging to the orbits A, B, C, D

(e.g., $8A_0 + 4B_1 + 2C_2 + D_3$ or $8B_2 + 4C_1 + 2D_0 + A_0$ are elements of (A, B, C, D)). In the next theorem, we show all the classes, which provide normal additive magic squares.

Remark 1. We obtain normal additive magic squares, which corresponds to all the normal multiplicative magic squares, only from the classes

$$(A, C, D, E)$$

$$(B, B, C, C)$$

$$(B, B, C, D)$$

$$(B, B, D, D)$$

$$(B, C, C, D)$$

$$(B, C, D, D)$$

$$(C, C, D, D).$$

We have to made clear some details

(1) When we choose a form in the orbit C, the forms available in the orbit D are only two and vice versa, except for the class (C, C, D, D).

(2) When we have a class with two forms from the same orbits, their sum must not be $\mathbb{U}$.

(3) We can not take two times the same form in the same class.

Considering this remarks, we can count the magic squares obtainable from these classes and we check that they are exactly 4224.

For the class (A, C, D, E), we can choose 2 forms from the orbit A, 4 from the orbit C, only 2 from the orbit D, and 2 from E. Thus from this class, we can obtain 32 normal additive magic squares, unless of permutations. Thus, we have $32 \cdot 24 = 768$ normal additive magic squares from (A, C, D, E), and we write

$$|(A, C, D, E)| = 768.$$

Similarly, we find

$$|(B, B, C, C)| = (4 \cdot 2 \cdot 4 \cdot 2) \cdot 6 = 384,$$

$$|(B, B, C, D)| = (4 \cdot 2 \cdot 4 \cdot 2) \cdot 12 = 768,$$

$$|(B, B, D, D)| = (4 \cdot 2 \cdot 4 \cdot 2) \cdot 6 = 384,$$

$$|(B, C, C, D)| = (4 \cdot 4 \cdot 2 \cdot 2) \cdot 12 = 768,$$

$$|(B, C, D, D)| = (4 \cdot 2 \cdot 4 \cdot 2) \cdot 12 = 768,$$

$$|(C, C, D, D)| = (4 \cdot 2 \cdot 4 \cdot 2) \cdot 6 = 384,$$

and

$$768 + 384 + 768 + 384 + 768 + 768 + 384 = 4224.$$

Remark 2. All the magic squares that can be represented through our notation can be classified and identified by the membership class.

Remark 3. Such representation allows to construct in a simple way all the normal multiplicative magic squares of order 4.

We conclude this paper with a further example.

Example 3. Let us consider the factorization $2 \cdot 3 \cdot 5 \cdot 67$ of 2010. We take a magic square in $\mathbb{M}_{2010,4}$

$$\begin{pmatrix} 2010 & 5 & 67 & 6 \\ 3 & 134 & 10 & 1005 \\ 2 & 201 & 15 & 670 \\ 335 & 30 & 402 & 1 \end{pmatrix} = \begin{pmatrix} 2{\cdot}3{\cdot}5{\cdot}67 & 5 & 67 & 2{\cdot}3 \\ 3 & 2{\cdot}67 & 2{\cdot}5 & 3{\cdot}5{\cdot}67 \\ 2 & 3{\cdot}67 & 3{\cdot}5 & 2{\cdot}5{\cdot}67 \\ 5{\cdot}67 & 2{\cdot}3{\cdot}5 & 2{\cdot}3{\cdot}67 & 1 \end{pmatrix},$$

its image through f is

$$\begin{pmatrix} 1111 & 0010 & 0001 & 1100 \\ 0100 & 1001 & 1010 & 0111 \\ 1000 & 0101 & 0110 & 1011 \\ 0011 & 1110 & 1101 & 0000 \end{pmatrix} = \begin{pmatrix} 15 & 2 & 1 & 12 \\ 4 & 9 & 10 & 7 \\ 8 & 5 & 6 & 11 \\ 3 & 14 & 13 & 0 \end{pmatrix}.$$

This square belongs to the class (C, C, D, D), in fact it can be decomposed as follows:

$$8\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$+ 2\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

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